New Results on the Exponent Set of Primitive Nearly Reducible Matrices

Li Bing

Department of Mathematics
University of Science and Technology of China
Hefei, Anhui, China

Submitted by Richard A. Brualdi

ABSTRACT

For a primitive, nearly reducible matrix, J. A. Ross defined $e(n)$ to be the least integer greater than 5 that no $n \times n$ matrix has as its exponent, and he proved

$$n+2 \leqslant e(n) \leqslant n^2-4n+6.$$

Recently, J. Y. Shao proved

$$e(n) \geqslant \frac{n^2-2n+10}{9+1},$$

and Yang Shangjun and J. P. Barker proved

$$e(n) > \frac{n^2}{4} - \left(\frac{n}{2}\right)^{3/2},$$

if a certain conjecture in number theory is true. In this paper we obtain the following new results:

$$e(n) \geqslant \left[\frac{n}{2}\right]\left(\left[\frac{n}{2}\right]+1\right)+1$$

for $n \geqslant 5$, and

$$e(n) \geqslant m^2\left[\frac{n}{m+1}\right]^2,$$

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where m is any given positive integer and n is large enough. In fact we have determined $e(n)$ asymptotically:

$$\lim_{n \to \infty} \frac{e(n)}{n^2} = 1.$$

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1. INTRODUCTION

In this paper we investigate the properties of the exponent set of primitive, nearly reducible matrices. There have been many contributions to this interesting subject. In 1980 R. A. Brualdi and J. A. Ross first studied it and gave some important results [1, 2]. Ross [2] defined $e(n)$ to be the least integer greater than 5 that no $n \times n$ primitive, nearly reducible matrix has as its exponent. Finding $e(n)$ for each $n \geqslant 5$ is difficult. It is the first open problem of Ross [2]. Recently there have been some results on this problem.

In [6], J. Y. Shao proved

$$e(n) \geqslant \frac{n^2 - 2n + 10}{9+1}.$$

In [7], Yang Shangjun and George P. Barker proved

$$e(n) > (p+1)(n-p) \tag{1.1}$$

for $n \geqslant 2p+1$, where $p \geqslant 11$ is a prime less than 100,000; and if a certain conjecture about the consecutive primes in the number theory is true, then (1.1) holds and

$$e(n) \geqslant \frac{n^2}{4} - \left(\frac{n}{2}\right)^{3/2}.$$

We give two better estimates for $e(n)$:

$$e(n) \geqslant \left[\frac{n}{2}\right]\left(\left[\frac{n}{2}\right] + 1\right) + 1 \tag{I}$$

for any $n \geqslant 5$, and

$$e(n) \geqslant m^2 \left[\frac{n}{m+1} \right]^2, \tag{II}$$

where m is any given positive integer and n is large enough. The lower bound in (II) determines $e(n)$ asymptotically. In fact, we have showed

$$\lim_{n \to \infty} \frac{e(n)}{n^2} = 1.$$

In this paper, we always use the same notations and definitions as in [1, 2, 6]. For an $n \times n$ nonnegative matrix A, we define its associated digraph $D(A) = (V, E)$, where $V = \{1, 2, 3, \dots, n\}$, $E = \{(i, j) | a_{i,j} > 0\}$. Obviously $D(A)$ depends only on the zero-nonzero pattern of A, and it is well known that A is nearly reducible iff $D(A)$ is minimally strong. We use graph theoretical language to formulate and prove our results.

A graph D is primitive iff there exists an integer $k > 0$ such that there is a walk from i to j with length k in D for any ordered pair of vertices $i, j \in V(D)$. The least such k is called the exponent of D and denoted by $\gamma(D)$. We define $\gamma(A) = \gamma(D(A))$. The following characterization is very useful.

THEOREM 1.1 [4, pp. 49–50]. *Suppose D is a strong digraph in which the circuit lengths are $r_1, r_2, \dots, r_k$. Then D is primitive if and only if* $\gcd(r_1, r_2, \dots, r_k) = 1$.

We use the following definitions:

DEFINITION 1.1. Suppose D is a primitive digraph in which the circuit lengths are $r_1, r_2, \dots, r_k$. Let (i, j) be an ordered pair of vertices in $V(D)$. We define the nonnegative integer $d(i, j)$ as the length of the shortest path from i to j which has at least one vertex on some circuit of length r_u for each $u = 1, 2, \dots, k$.

If $i = j$ and if for each $u = 1, 2, \dots, k$ there is some circuit of length r_u through vertex i, then $d(i, i) = 0$.

DEFINITION 1.2. Suppose D is a primitive digraph and (i, j) is an ordered pair of vertices in $V(D)$. We also define $\gamma(i, j)$ as the least integer such that for any integer $m \geqslant \gamma(i, j)$ there is a path from i to j with length m.

It is obvious that

$$\gamma(D) = \max\{\gamma(i,j) \mid i,j \in V(D)\}.$$

Suppose $a_1, a_2, \ldots, a_k$ are distinct positive integers and $\gcd(a_1, a_2, \ldots, a_k) = 1$. The Frobenius number $\Phi(a_1, a_2, \ldots, a_k)$ is the least integer such that any integer m greater than or equal to it can be expressed as

$$m = c_1 a_1 + c_2 a_2 + \cdots + c_k a_k$$

where $c_1, c_2, \ldots, c_k$ are nonnegative integers. $\Phi(a_1, a_2, \ldots, a_k)$ is finite if and only if

$$\gcd(a_1, a_2, \ldots, a_k) = 1.$$

If $k = 2$, then

$$(a_1, a_2) = (a_1 - 1)(a_2 - 1) = a_1 a_2 - a_1 - a_2 + 1.$$

THEOREM 1.2 [5]. *If D is a primitive digraph in which the circuit lengths are $r_1, r_2, \ldots, r_k$, then $\gamma(i,j) \leqslant d(i,j) + \Phi(r_1, r_2, \ldots, r_k)$ for any vertices i, j in $V(D)$.*

From Theorem 1.1, the digraphs in Lemma 2.5 are all primitive. Theorem 1.2 will be used to compute the exponents of all the digraphs in this paper.

2. SOME BASIC LEMMAS

We define $\mathrm{NE}(n)$ as the exponent set of primitive, nearly reducible $n \times n$ matrices. Obviously $\mathrm{NE}(n)$ is also the exponent set of primitive, minimally strong digraphs with n vertices.

First we note the following useful lemma.

LEMMA 2.1 [6, Lemma 2.3]. $\mathrm{NE}(n) \subset \mathrm{NE}(n+1)$ *for* $n = 5, 6, 7, \ldots$.

From this lemma we get the following corollary.

COROLLARY. $e(n) \leqslant e(n+1)$ *for* $n = 5, 6, 7, \ldots$.

Now we list some results about $e(n)$ for the small n. These results can be found in [7, Propositions 2.2, 2.3; Table 1].

LEMMA 2.2. *$e(5)=7$, $e(6)=13$, $e(7)=20$, $e(8)=21$, and for*

$$\begin{array}{l} n=10,11,12,13,14,\ 15,\ 16,\ 17,\ 18,\ 19,\ 20,\ 21,\ 22,\ 23,\\ e(n)\geqslant 36,47,49,63,87,106,107,126,151,176,203,232,233,248. \end{array}$$

LEMMA 2.3.

$$e(n)\geqslant\left[\frac{n}{2}\right]\left(\left[\frac{n}{2}\right]+1\right)+1$$

for $n=5,6,7,\ldots,28$.

Proof. When $n=5,6,\ldots,23$, we use Lemma 2.2. When $n=24,\ldots,28$,

$$e(n)\geqslant e(23)\geqslant 248>\left[\frac{28}{2}\right]\left(\left[\frac{28}{2}\right]+1\right)+1>\left[\frac{n}{2}\right]\left(\left[\frac{n}{2}\right]+1\right)+1. \quad\blacksquare$$

We now prove two important lemmas. We define $p(n)$ to be the nth prime number.

LEMMA 2.4. *If $n\geqslant 4$, then*

$$p(1)p(2)\cdots p(n)>p(n+1)^2.$$

Proof. We use the induction on n.

(1) $n=4$:

$$p(1)p(2)p(3)p(4)=2\times3\times5\times7=210>11\times11=[p(5)]^2.$$

(2) Suppose $p(1)p(2)\cdots p(n)>[p(n+1)]^2$ for $n\geqslant 4$.

(3) First, there is at least one prime between m and $2m$ for any nonnegative integer m [8, Chapter 5]. Therefore we have

$$p(i+1)<2p(i)$$

for any $i\geqslant 1$; then

$$\begin{aligned} p(1)p(2)\cdots p(n)p(n+1)&>[p(n+1)]^2p(n+1)\\ &>\left(\frac{p(n+2)}{2}\right)^2p(n+1)>[p(n+2)]^2. \quad\blacksquare \end{aligned}$$

LEMMA 2.5. *Suppose s, t are integers, $2 < s < t$, $\gcd(s,t) = 1$, and*

$$t \not\equiv -1 \pmod{s}. \tag{2.1}$$

Let $n \geqslant \max\{s+t-1, [t/s]+t+1\}$, and let

$$f(s,t) = st - s + 1, \qquad F(s,t) = st + t - s - 1.$$

Then

$$[f(s,t), F(s,t)] \subset \mathrm{NE}(n),$$

where $[x,y] = \{i \text{ integer} : x \leqslant i \leqslant y\}$.

Proof. We construct some primitive, minimally strong digraphs with exponents $F(s,t), F(s,t)-1, \ldots, G(s,t)+1, G(s,t), G(s,t)-1, \ldots, f(s,t)$, where $G(s,t) = st + t - 2s - 1$. By Lemma 2.1, the number of vertices in each graph can be less than n. Any elementary circuit in these digraphs has length s or t.

Figure 1: *The digraph D with exponent $F(s,t)$.* The number of vertices in this digraph is $s+t-1$. By Theorem 1.2, the exponent of D is

$$\begin{aligned}\gamma(D) &= \gamma(s+1, s+t-1) = d(s+1, s+t-1) + \Phi(s,t) \\ &= st + t - s - 1 = F(s,t).\end{aligned}$$

Figure 2: *The digraph with exponent $F(s,t)-1$.* The number of vertices of this graph is $s+t-2$. By Theorem 1.2, the exponent of this digraph is

$$\gamma(D) = \gamma(s+1, s+t-2) = 2t - 4 + \Phi(s,t) = st + t - s - 2 = F(s,t) - 1.$$

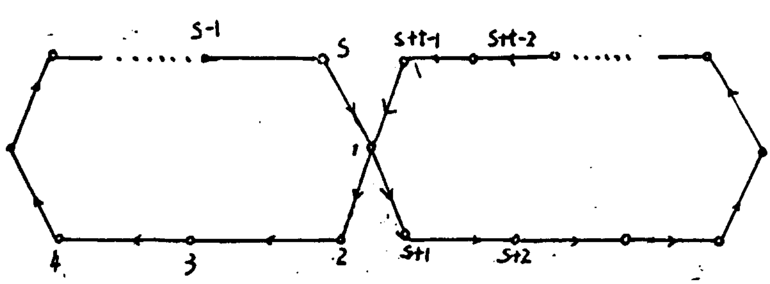

FIG. 1.

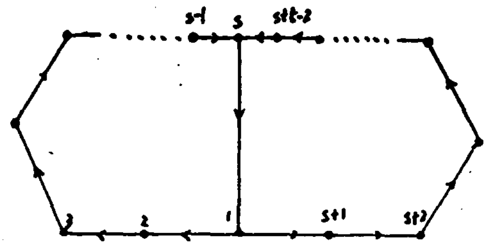

FIG. 2.

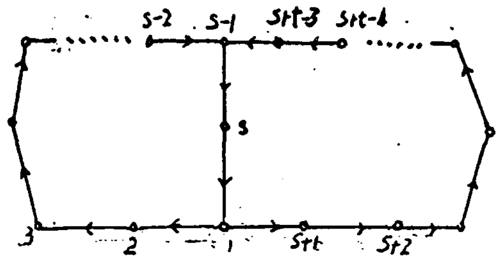

FIG. 3.

Figure 3: *The digraph with exponent* $F(s,t)-2$. The number of vertices in D is $s+t-3$, and the exponent is

$$\gamma(D)=\gamma(s+1,s+t-3)=d(s+1,s+t-3)+\Phi(s,t)$$

$$=t+t-4+st-s-t+1=F(s,t)-2.$$

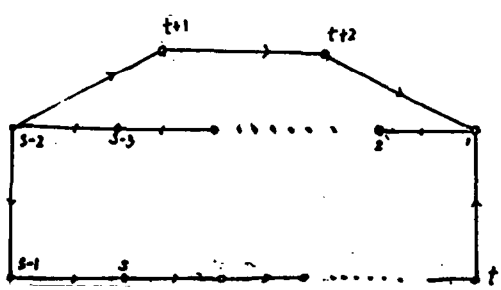

FIG. 4.

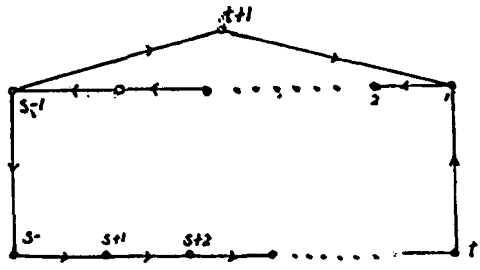

FIG. 5.

Figure 4: *The digraph with exponent* $G(s,t)+1$. The number of vertices of this graph is $t+2$, and the exponent is

$$\gamma(D)=\gamma(s-1,t)=2t-s+1+\Phi(s,t)=st+t-2s+2=G(s,t)+1.$$

Figure 5: *The digraph with exponent* $G(s,t)$. The number of vertices of this graph is $t+1$, and the exponent is

$$\gamma(D)=\gamma(s,t)=2t-s+\Phi(s,t)=st+t-2s+1=G(s,t).$$

Figure 6: *The digraph with exponent* $G(s,t)-1$. The number of vertices of this digraph is $t+2$. The exponent is

$$\gamma(D)=d(s+1,t)+\Phi(s,t)=2t-s-1+st-s-t+1=G(s,t)-1.$$

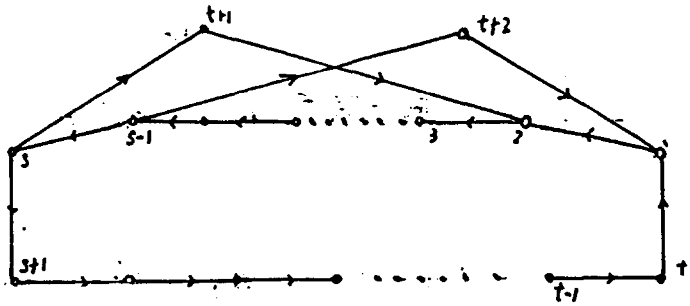

FIG. 6.

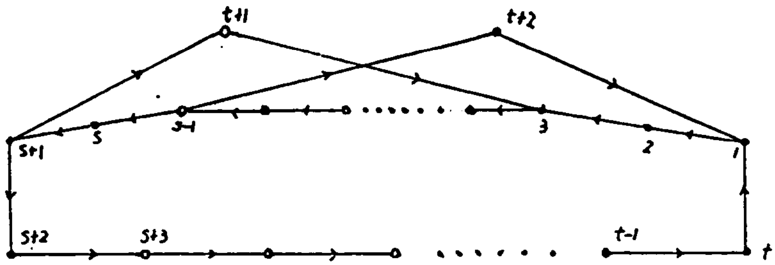

FIG. 7.

Figure 7: *The digraph with exponent $G(s,t)-2$.* The number of vertices of this graph is $t+2$, and the exponent is

$$\gamma(D)=\gamma(s+2,t)=d(s+2,t)+\Phi(s,t)$$
$$=t-s-2+t+st-s-t+1=G(s,t)-2.$$

Figure 8: *The digraph with exponent $f(s,t)$.* There are only a circuit with length t and $[t/s]+1$ circuits with length s. Clearly this graph is

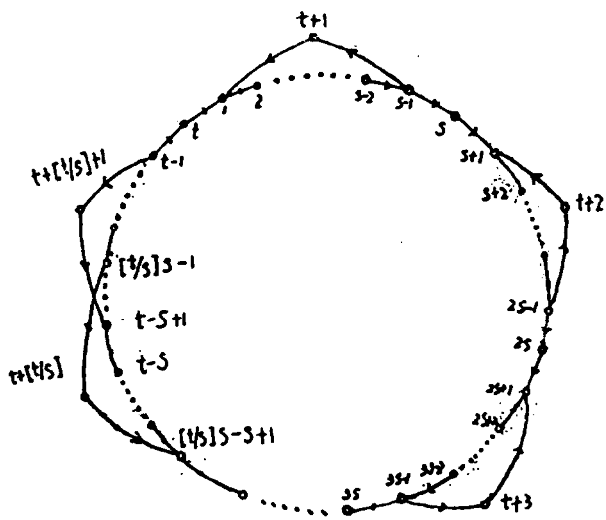

FIG. 8.

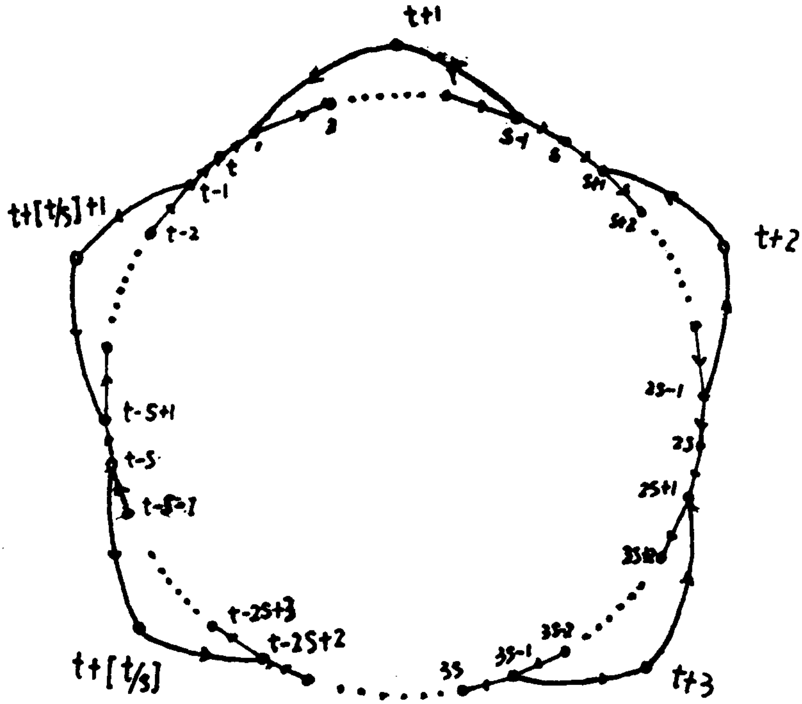

FIG. 9.

primitive and nearly reducible. The number of vertices is $t+[t/s]+1$. We can compute the exponent of D as before:

$$\gamma(D)=d(t,t)+\Phi(s,t)=t+st-s-t+1=f(s,t). \qquad \blacksquare$$

LEMMA 2.6. *Suppose* s,t *are integers,* $2<s<t$, $\gcd(s,t)=1$, *and* $t\equiv -1 \pmod{s}$. *Let* $n\geqslant \max(s+t-1, t+[t/s]+1\}$. *Then*

$$[f(s,t)+1, F(s,t)]\subset \mathrm{NE}(n).$$

Proof. We consider the last digraph in Lemma 2.5. In this case, the exponent of this graph is $f(s,t)+1$. (See Figure 9.)

The number of vertices is $t+[t/s]+1$, and

$$\begin{aligned}\gamma(D)&=d(t+[t/s]+1,t+[t/s])+\Phi(s,t)\\&=t+1+st-s-t+1=f(s,t)+1.\end{aligned} \qquad \blacksquare$$

3. THE GENERAL RESULT FOR $e(n)$

We have proved

$$e(n) \geqslant \left[\frac{n}{2}\right]\left(\left[\frac{n}{2}\right]+1\right)+1$$

for the small n. Now we prove it for any $n \geqslant 5$.

THEOREM 3.1. $[n^2+2n+1, n^2+3n+2] \subset \mathrm{NE}(2n+2)$ *for each* $n \geqslant 4$.

Proof. Let $s = n+1$ and $t = n+2$. Then (2.1) is true and

$$\gcd(s,t)=1,$$

$$\max\{s+t-1, t+[t/s]+1\} = \max\{2n+2, n+4\} = 2n+2.$$

So the conditions in Lemma 2.5 are satisfied, and

$$[f(s,t), F(s,t)] \subset \mathrm{NE}(2n+2).$$

We easily have

$$F(n+1,n+2) = (n+1)(n+2)+n+2-(n+1)-1 = n^2+3n+2,$$

$$f(n+1,n+2) = (n+1)(n+2)-(n+1)+1 = n^2+2n+2,$$

and therefore

$$[n^2+2n+2, n^2+3n+2] \subset \mathrm{NE}(2n+2). \qquad \blacksquare$$

THEOREM 3.2. *If* $n > 4$ *is odd, then*

$$[n^2+n+1, n^2+2n+1] \subset \mathrm{NE}(2n+2).$$

Proof. Let $s = n$ and $t = n+2$. Because n is odd, $\gcd(n, n+2) = 1$. Clearly (2.1) is true, and $2n+2$ is greater than $s+t-1$ and $t+[t/s]+1$. Furthermore, $f(n,n+2) = n^2+n+1$ and $F(n,n+2) = n^2+2n+1$. By Lemma 2.5, then

$$[n^2+n+1, n^2+2n+1] \subset \mathrm{NE}(2n+2). \qquad \blacksquare$$

THEOREM 3.3. *If $n \geqslant 8$ is even, then*

$$[n^2 + n - 1, n^2 + 2n] \subset \mathrm{NE}(2n+2).$$

Proof. Let $s = n - 1$ and $t = n + 3$. Then $\gcd(s,t) = 1$. Equation (2.1) holds because $n \geqslant 8$. We have $F(n-1, n+3) = n^2 + n - 1$ and $f(n-1, n+3) = n^2 + 2n$, by Lemma 2.5. ∎

THEOREM 3.4. *Suppose $n \geqslant 12$. Then*

$$n^2 + 2n + 1 \in \mathrm{NE}(2n+2).$$

Proof. Suppose P is the least prime which is not the divisor of $2n+3$. Let

$$L = \frac{P+1}{2}.$$

Then:

(1) We have

$$\begin{aligned}
\gcd(n - L + 2, n + L + 1) &= \gcd(n - L + 2, n + L + 1 - (n - L + 2)) \\
&= \gcd(n - L + 2, 2L - 1) \\
&= \gcd(2(n - L + 2) + 2L - 1, 2L - 1) \\
&= \gcd\left(2n + 3, 2x\frac{P+1}{2} - 1\right) \\
&= \gcd(2n + 3, P) = 1.
\end{aligned}$$

(2) We prove that

$$n + 3L - L^2 - 1 > 0.$$

Case 1. When $P = 3, 5$, or 7, then $2 \leqslant L \leqslant 4$. Hence

$$n + 3L - L^2 - 1 > n - \left(L - \tfrac{3}{2}\right)^2 \geqslant 12 - \left(\tfrac{5}{2}\right)^2 > 0.$$

Case 2. When $P > 7$, let P be the ith prime $p(i)$, so that $i \geqslant 5$. From the definition of P, we know $p(2), p(3), \dots, p(i-1)$ are the divisors of $2n+3$, then

$$\begin{aligned} 4(n+3L-L^2-1) &= 4\left(n+3\frac{P+1}{2}-\frac{(P+1)^2}{4}-1\right) \\ &> 2(2n+3)-P^2 \\ &\geqslant 2p(2)p(3)\cdots p(i-1)-p(i)^2. \end{aligned}$$

Using Lemma 2.4, we have

$$n+3L-L^2-1 > \frac{p(1)p(2)p(3)\cdots p(i-1)-p(i)^2}{4} > 0.$$

(3) Let $s = n - L + 2$, $t = n + L + 1$. We know $L < n/2$ in (2). Then

$$\left[\frac{t}{s}\right]+1 \leqslant \left[\frac{3n/2+1}{n/2+2}\right]+1 < 4$$

and

$$2n+2 \geqslant \max\{s+t-1, [t/s]+t+1\}.$$

We easily have

$$F(s,t) = n^2+2n+1+(n+3L-L^2-1),$$

$$f(s,t) = n^2+2n+1+2L-L^2.$$

Now we prove $n^2+2n+1 \in \mathrm{NE}(2n+2)$ for $L = 2$ and $L > 2$.

If $L = 2$, then (2.1) holds for $n - L + 2$ and $n + L + 1$. By (1) and (3), we know the conditions of Lemma 2.5 are all satisfied. It is clearly that $n^2 + n + 1 = f(s,t)$ and $n^2+2n+1 \in \mathrm{NE}(2n+2)$.

If $L > 2$, then $1+2L-L^2 \leqslant 0$ and $n^2+n+1 \geqslant f(s,t)+1$. By (2), we have

$$F(s,t) \geqslant n^2+2n+1.$$

Hence n^2+2n+1 is an element of $[f(s,t)+1, F(s,t)]$. By Lemma 2.6, this means

$$n^2 + 2n + 1 \in \mathrm{NE}(2n+1). \qquad \blacksquare$$

THEOREM 3.5. *Suppose $n \geqslant 12$. Then*

$$[n(n+1)+1, (n+1)(n+2)] \subset \mathrm{NE}(2n+2). \tag{3.1}$$

Proof. When n is odd, we use Theorem 3.1 and Theorem 3.2. When n is even, we use Theorem 3.1, Theorem, 3.3 and Theorem 3.4 ■

THEOREM 3.6. *Suppose $n \geqslant 3$. Then*

$$e(2n) \geqslant n(n+1)+1. \tag{3.2}$$

Proof. We prove it by induction on n.
When $n = 3, 4, 5, \ldots, 14$, (3.2) holds by Lemma 2.3. Suppose

$$e(2n) \geqslant n(n+1)+1 \tag{3.3}$$

for $n \geqslant 14$. By the definition of $e(n)$ and Lemma 2.1, (3.1) and (3.3) show

$$e(2n+2) \geqslant (n+1)(n+2)+1. \qquad \blacksquare$$

THEOREM 3.7.

$$e(n) \geqslant \left[\frac{n}{2}\right]\left(\left[\frac{n}{2}\right]+1\right)+1$$

for $n \geqslant 5$.

Proof. By the corollary of Lemma 2.1, $e(n) \leqslant e(n+1)$ and $e(5) = 7$. ■

4. THE LOWER BOUND OF $e(n)$ FOR n LARGE

Now we prove

$$e((m+1)n) \geqslant m^2 n^2 + 1$$

for any given positive integer m and n large enough. First we improve Lemma 2.5.

LEMMA 4.1. *Suppose* s, t *are integers with* $2 < s < t$, $\gcd(s,t)=1$, *and* $n \geqslant [t/s]+t+1$. *Let*

$$F(s,t,n)=\begin{cases} st+t-s-1 & \textit{if} \quad n \geqslant s+t-1, \\ st-2s+n & \textit{if} \quad n < s+t-1, \end{cases}$$

$$f(s,t)=st-s+1.$$

Then

$$[f(s,t)+1, F(s,t,n)] \subset \mathrm{NE}(n).$$

Proof. If $n \geqslant s+t-1$, we use Lemma 2.6.

If $n < s+t-1$, we consider the digraphs in Lemma 2.6 with exponent less than or equal to $st-2s+n$ and easily find the numbers of their vertices are equal to or less than n. This shows our claim by Lemma 2.1. ■

Now we offer a useful lemma. This lemma is a corollary to the prime numbers theorem in [8].

LEMMA 4.2 [8, Chapter 5]. *Suppose* $a > 1$ *is a constant. Then there exists at least three primes in* $[b, ab]$ *if* b *is large enough.*

THEOREM 4.3. *Suppose* $m > 2$ *is an integer and* n *is large enough. Then*

$$\left[m^2n^2+2, m^2(n+1)^2+1\right] \subset \mathrm{NE}((m+1)(n+1)). \tag{4.1}$$

Proof. Let $s_k = mn+k$, $t_k = mn+k+1$ for $k=0,1,2,\ldots,m-1,m$. By Lemma 4.1, we have

$$[f(s_k,t_k)+1, F(s_k,t_k,(m+1)(n+1))] \subset \mathrm{NE}((m+1)(n+1)). \tag{4.2}$$

Because $f(s_0,t_0)=m^2n^2+1$ and $f(s_m,t_m)=m^2(n+1)^2+1$, (4.1) is true if and only if

$$[F(s_k,t_k,(m+1)(n+1))+1, f(s_{k+1},t_{k+1})] \subset \mathrm{NE}((m+1)(n+1)). \tag{4.3}$$

Now we prove (4.3). First, let

$$a_i=\begin{cases} 0 & \text{if} \quad i=0,1,2,\ldots,4m-3; \\ 3k-m+1 & \text{if} \quad i=4m-2, \end{cases}$$

and

$$b_i = [in/2] + a_i, \qquad i = 0,1,2,\ldots,4m-3,4m-2.$$

We easily have

$$b_0 = 0,$$
$$F(s_k, t_k, (m+1)(n+1)) + b_{4m-2} = f(s_{k+1}, t_{k+1}).$$

Then (4.3) implies

$$[F(s_k, t_k, (m+1)(n+1)) + b_i + 1, F(s_k, t_k, (m+1)(n+1)) + b_{i+1}]$$
$$\subset \mathrm{NE}((m+1)(n+1)) \qquad \text{for each} \quad i = 0,1,2,\ldots,4m-3. \tag{4.4}$$

We consider the set

$$\left[1 + 2\sqrt{(2m-1)n - b_i + 3k - m + 1}, 2 + 2\sqrt{2mn - b_{i+1} + 2k + \tfrac{1}{4}}\right].$$

That is the set

$$\Big[1 + 2\sqrt{(2m-1)n - [in/2] - a_i + 3k - m + 1},$$
$$2 + 2\sqrt{2mn - [(i+1)n/2] - a_{i+1} + 2k + \tfrac{1}{4}}\Big].$$

By Lemma 4.2, there exist at least three primes in this set, say a, b, c, when n is large enough. If all of them are the divisors of $2mn + 2k + 3$, then

$$2mn + 2k + 3 \geqslant abc > \left(1 + 2\sqrt{(2m-1)n - [in/2] - a_i + 3k - m + 1}\right)^3.$$

This is impossible when n is large enough. Hence at least one of them is not a divisor of $2mn + 2k + 3$. Suppose it is c. Clearly c is odd. Let $d = (c-1)/2$ and

$$\tilde{s} = mn + k + 1 - d, \qquad \tilde{t} = mn + k + 2 + d.$$

Then:

(1) From the definition of d, $d < n/2$ when n is large enough. Then

$$(m+1)(n+1) \geqslant \tilde{t} + [\tilde{t}/\tilde{s}] + 1.$$

(2) We have

$$\begin{aligned}\gcd(\tilde{s},\tilde{t}) &= \gcd(\tilde{s},\tilde{t}-\tilde{s})\\ &= \gcd(mn+k+1-d,2d+1)\\ &= \gcd(2(mn+k+1-d)+2d+1,2d+1)\\ &= \gcd(2mn+2k+3,2d+1)\\ &= \gcd(2mn+2k+3,c)=1,\end{aligned}$$

because c is a prime which is not the divisor of $2mn+2k+3$.

(3) From the definition of d, we know

$$\sqrt{(2m-1)n-b_i+3k-m+1} \leqslant d \leqslant \tfrac{1}{2}+\sqrt{2mn-b_{i+1}+2k+\tfrac{1}{4}}\,.$$

Hence

$$\begin{aligned}&f(\tilde{s},\tilde{t})+1\\ &\quad= \tilde{s}\tilde{t}-\tilde{s}+2\\ &\quad= (mn+k+1-d)(mn+k+2+d)-(mn+k+1-d)+2\\ &\quad= m^2n^2+k^2+2mn+2kmn+2k+3-d^2\\ &\quad\leqslant m^2n^2+k^2+2kmn+2mn+2k+3-\{(2m-1)n-b_i+3k-m+1\}\\ &\quad\leqslant F(s_k,t_k,(m+1)(n+1))+b_i+1,\end{aligned}$$

$$\begin{aligned}&F(\tilde{s},\tilde{t},(m+1)(n+1))\\ &\quad= \tilde{s}\tilde{t}-2\tilde{s}+(m+1)(n+1)\\ &\quad= (mn+k+1-d)(mn+k+2+d)\\ &\qquad -2(mn+k+1-d)+(m+1)(n+1)\\ &\quad= m^2n^2+(2k+2)mn+n+k^2+k+m+\tfrac{1}{4}-\left(d-\tfrac{1}{2}\right)^2\\ &\quad\geqslant m^2n^2+(2k+2)mn+n+k^2+k+m+\tfrac{1}{4}-\left(2mn-b_{i+1}+2k+\tfrac{1}{4}\right)\\ &\quad\geqslant m^2n^2+2kmn+n+k^2-k+m+1+b_{i+1}\\ &\quad\geqslant F(s_k,t_k,(m+1)(n+1))+b_{i+1}.\end{aligned}$$

These mean that $[F(s_k,t_k,(m+1)(n+1))+b_i+1, F(s_k,t_k,(m+1)(n+1))+b_{i+1}]$ is a subset of $[f(\bar{s},\bar{t})+1, F(\bar{s},\bar{t},(m+1)(n+1))]$. By Lemma 4.1, we know (4.3) holds for each $i=0,1,2,\dots,4m-3$. ■

THEOREM 4.4. *$e((m+1)n)\geqslant m^2n^2+1$ when n is large enough.*

Proof. By Theorem 4.3, there exists a positive integer N such that $[m^2n^2+2, m^2(n+1)^2+1]\subset \mathrm{NE}((m+1)(n+1))$ for any $n\geqslant N$. Then

$$\left[m^2N^2+2, m^2(N+1)^2+1\right]\subset \mathrm{NE}((m+1)(N+1)),$$

$$\left[m^2(N+1)^2+2, m^2(N+2)^2+1\right]\subset \mathrm{NE}((m+1)(N+2)),$$

$$\left[m^2(N+2)^2+2, m^2(N+3)^2+1\right]\subset \mathrm{NE}((m+1)(N+3)),$$

$$\vdots$$

$$\left[m^2(n-1)^2+2, m^2(n-1)^2+1\right]\subset \mathrm{NE}((m+1)n),$$

$$\left[m^2(n-1)^2+2, m^2n^2+1\right]\subset \mathrm{NE}((m+1)(n+1)).$$

We claim, by Lemma 2.1, the union of these sets is contained in $\mathrm{NE}((m+1)(n+1))$. That is,

$$[m^2N^2+2, m^2n^2+1]\subset \mathrm{NE}((m+1)(n+1)). \tag{4.5}$$

Suppose $n>2N$. By Theorem 3.6,

$$e((m+1)n)>e((m+1)2N)>(m+1)N\{(m+1)N+1\}>m^2N^2+2. \tag{4.6}$$

By the definition of $e(n)$ and (4.5),(4.6), we have

$$e((m+1)n)>m^2n^2+1$$

when n is greater than $2N$. ■

By Lemma 2.1 and its corollary, we easily have a lower bound of $e(n)$.

THEOREM 4.5.

$$e(n) \geqslant m^2 \left[\frac{n}{(m+1)} \right]^2 + 1$$

when n is large enough.

By Theorem 4.5 and the Ross's estimate of $e(n)$ [2], $e(n) \leqslant n^2 - 4n + 6$, we can determine $e(n)$ asymptotically. That is our final theorem.

THEOREM 4.6.

$$\lim_{n \to \infty} \frac{e(n)}{n^2} = 1.$$

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